FINITE DIMENSIONAL HOPF ALGEBRAS OVER THE DUAL GROUP ALGEBRA OF THE SYMMETRIC GROUP IN THREE LETTERS

NICOLÁS ANDRUSKIEWITSCH AND CRISTIAN VAY

Dedicated to Mia Cohen with affection

Abstract. We classify finite-dimensional Hopf algebras whose coradical is isomorphic to the algebra of functions on $\mathbb{S}_3$. We describe a new infinite family of Hopf algebras of dimension 72.

1. Introduction

This work is a contribution to the classification of finite-dimensional Hopf algebras over an algebraically closed field $\Bbbk$ of characteristic 0, problem posed by I. Kaplansky in 1975. We are specifically interested in Hopf algebras whose coradical is the dual of a non-abelian group algebra. We note that there are very few classification results on finite-dimensional Hopf algebras whose coradical is a Hopf subalgebra but not a group algebra, a remarkable exception being [CDMM].

Let A be a Hopf algebra whose coradical H is a Hopf subalgebra. It is well-known that the associated graded Hopf algebra of A is isomorphic to $R\#H$ where $R = \oplus_{n\in\mathbb{N}_0} R^n$ is a braided Hopf algebra in the category ${}^H_H\mathcal{YD}$ of Yetter-Drinfield modules over H. As explained in [AnS2], to classify finite-dimensional Hopf algebras A whose coradical is isomorphic to H we have to deal with the following questions:

(a) Determine all Yetter-Drinfield modules V over H such that the Nichols algebra $\mathfrak{B}(V)$ has finite dimension; find an efficient set of relations for $\mathfrak{B}(V)$.
(b) If $R = \oplus_{n\in\mathbb{N}_0} R^n$ is a finite-dimensional Hopf algebra in ${}^H_H\mathcal{YD}$ with $V = R^1$, decide if $R \simeq \mathfrak{B}(V)$ (generation in degree one).
(c) Given V as in (a), classify all A such that $\operatorname{gr} A \simeq \mathfrak{B}(V)\#H$ (lifting).

Now the category ${}^{H^*}_{H^*}\mathcal{YD}$ is braided equivalent to ${}^H_H\mathcal{YD}$ see e.g. [AG1, 2.2.1]. Therefore, the answers to the questions (a) and (b) in ${}^{H^*}_{H^*}\mathcal{YD}$ give the

Date: February 1, 2011.
2000 *Mathematics Subject Classification.* 16W30.
This work was partially supported by ANPCyT-Foncyt, CONICET, Ministerio de Ciencia y Tecnología (Córdoba) and Secyt (UNC). Part of the work of C. V. was done as a fellow of the Erasmus Mundus programme of the EU in the University of Antwerp. He thanks to Prof. Fred Van Oystaeyen for his warm hospitality and help.

analogous answers in ${}^H_H\mathcal{YD}$. Thus, if $H = \Bbbk^G$ is the algebra of functions on a finite group G, then we are reduced to know the solutions to questions (a) and (b) for the group algebra $\Bbbk G$. In the case $G = \mathbb{S}_3$, we know that

- there is a unique $V \in {}^{\Bbbk^{\mathbb{S}_3}}_{\Bbbk^{\mathbb{S}_3}}\mathcal{YD}$ such that $\dim \mathfrak{B}(V) < \infty$, namely $V = M((12), \mathrm{sgn})$ [AHS, Thm. 4.5].
- If $R = \oplus_{n\in\mathbb{N}_0} R^n$ is a finite-dimensional Hopf algebra in ${}^{\Bbbk^{\mathbb{S}_3}}_{\Bbbk^{\mathbb{S}_3}}\mathcal{YD}$, then $R \simeq \mathfrak{B}(V)$ [AG2, Thm. 2.1].

In this paper, we solve question (c) for $\Bbbk^{\mathbb{S}_3}$. We introduce a family of Hopf algebras $\mathcal{A}_{[a_1,a_2]}$, $(a_1, a_2) \in \Bbbk^2$; these are new as far as we know, except for $(a_1, a_2) = (0, 0)$. Let $\Gamma = \Bbbk^\times \times \mathbb{S}_3$, where $\Bbbk^\times$ is the group of the invertible elements of $\Bbbk$. We consider the right action $\triangleleft$ of the group on $\Bbbk^2$ defined by

$$(1.1)\quad (a_1, a_2)\triangleleft(\mu, (12)) = \mu(a_2, a_1), \qquad (a_1, a_2)\triangleleft(\mu, (123)) = -\mu(a_2, a_2-a_1).$$

We denote by $[a_1, a_2] \in \Gamma\backslash\Bbbk^2$ the equivalence class of (a_1, a_2) under this action; notice that $\Gamma\backslash\Bbbk^2$ is infinite. We prove in Theorem 3.5:

Main Theorem. *The set of isomorphism classes of finite-dimensional non-semisimple Hopf algebras with coradical $\Bbbk^{\mathbb{S}_3}$ is in bijective correspondence with $\Gamma\backslash\Bbbk^2$ via the assignment $[a_1, a_2] \leftrightsquigarrow \mathcal{A}_{[a_1,a_2]}$.*

To show that the Hopf algebras $\mathcal{A}_{[a_1,a_2]}$ have the right dimension we use the Diamond Lemma. To prove that any finite-dimensional non-semisimple Hopf algebra A with coradical $\Bbbk^{\mathbb{S}_3}$ belongs to this family, we describe the first term A_1 of the coradical filtration using [ArMŞ, Thm. 5.9.c)].

Acknowledgment. N. A. thanks Margaret Beattie for conversations on this problem in St. John's, May 2008.

2. Preliminaries

2.1. **Notation.** We fix an algebraically closed field $\Bbbk$ of characteristic zero. If V is a vector space, $T(V) = \oplus_{n\geq 0} V^{\otimes n}$ is the tensor algebra of V. If there is no place for confusion, we write $a_1 \cdots a_n$ instead of $a_1 \otimes \cdots \otimes a_n$ for $a_1, ..., a_n \in V$. If G is a group, we denote by e its identity element, by $\Bbbk G$ the group algebra of G and by $\Bbbk^G$ the dual group algebra, that is the algebra of functions on G. We denote by $\mathbb{S}_n$ the symmetric group on n letters. We use Sweedler notation but dropping the summation symbol. If H is an algebra, then we denote by ${}_H\mathcal{M}$, resp. ${}_H\mathcal{M}_H$, the category of left H-modules, resp. H-bimodules.

2.2. **The coradical filtration.** Let C be a coalgebra with comultiplication Δ. The coradical C_0 of C is the sum of all simple subcoalgebras of C. The coradical filtration of C is the family of subspaces defined inductively by $C_n := C_0 \wedge C_{n-1} = \Delta^{-1}(C \otimes C_0 + C_{n-1} \otimes C)$ for each $n \geq 1$. Then

$$(2.1)\qquad C_n \subseteq C_{n+1}, \quad C = \bigcup_{n\geq 0} C_n \quad \text{and} \quad \Delta(C_n) \subseteq \sum_{i=0}^{n} C_i \otimes C_{n-i},$$

for all $n \geq 0$ [M, Thm. 5.2.2]. We denote by $\operatorname{gr} C = \oplus_{n\geq 0} \operatorname{gr}^n C$ the associated graded coalgebra of C; as vector space $\operatorname{gr}^n C := C_n/C_{n-1}$ for all $n \geq 0$ where $C_{-1} = 0$. We denote by $G(C)$ the group of group-like elements of C. As usual, for $g, h \in G(C)$, $\mathcal{P}_{g,h}(C) = \{x \in C_1 | \Delta(x) = g \otimes x + x \otimes h\}$ is the space of (g,h)-skew primitive elements of C. If C is a bialgebra and $g = h = 1$, we write simply $\mathcal{P}(C)$ instead of $\mathcal{P}_{1,1}(C)$ and $x \in \mathcal{P}(C)$ is called a primitive element. A coalgebra E is called a *matrix coalgebra* of rank n if it has a basis $(e_{ij})_{1\leq i,j\leq n}$ such that the comultiplication and counit are defined by $\Delta(e_{ij}) = \sum_{k=1}^{n} e_{ik} \otimes e_{kj}$ and $\varepsilon(e_{ij}) = \delta_{ij}$ for all $1 \leq i, j \leq n$.

Lemma 2.1. *Let $D = \Bbbk g$ and $E = \langle e_{ij} | 1 \leq i, j \leq n\rangle$ be matrix coalgebras of rank 1 and n respectively. If $(x_i)_{i=1}^n \subset D \oplus E$ (direct sum of coalgebras) are such that*

$$\Delta(x_i) = x_i \otimes g + \sum_{j=1}^{n} e_{ij} \otimes x_j,$$

then there exist $a_1, ..., a_n \in \Bbbk$ such that $x_i = a_i g - \sum_{j=1}^{n} a_j e_{ij}$, $1 \leq i \leq n$.

Proof. Write $x_i = a_i g + \sum_{s,t=1}^{n} \alpha_{st}^i e_{st}$ for all $1 \leq i \leq n$, where $a_i, \alpha_{st}^i \in \Bbbk$, $1 \leq i, s, t \leq n$. Now we calculate $\Delta(x_i)$ in two ways:

$$\Delta(x_i) = \Delta\left(a_i g + \sum_{s,t=1}^{n} \alpha_{st}^i e_{st}\right) = a_i g \otimes g + \sum_{s,t,l=1}^{n} \alpha_{st}^i e_{sl} \otimes e_{lt} \qquad \text{and}$$

$$\begin{aligned}\Delta(x_i) &= \left(a_i g + \sum_{s,t=1}^{n} \alpha_{st}^i e_{st}\right) \otimes g + \sum_{j=1}^{n} e_{ij} \otimes \left(a_j g + \sum_{s,t=1}^{n} \alpha_{st}^j e_{st}\right) \\ &= a_i g \otimes g + \sum_{s,t=1}^{n} \alpha_{st}^i e_{st} \otimes g + \sum_{j=1}^{n} a_j e_{ij} \otimes g + \sum_{s,t,j=1}^{n} \alpha_{st}^j e_{ij} \otimes e_{st} \\ &= a_i g \otimes g + \sum_{\substack{s,t=1 \\ s\neq i}}^{n} \alpha_{st}^i e_{st} \otimes g + \sum_{t=1}^{n} (a_t + \alpha_{it}^i) e_{it} \otimes g + \sum_{s,t,j=1}^{n} \alpha_{st}^j e_{ij} \otimes e_{st}\end{aligned}$$

Then the second and third terms are zero. Thus $\alpha_{st}^i = 0$ and $\alpha_{it}^i = -a_t$, for $1 \leq i, s, t \leq n$, $s \neq i$. Hence $x_i = a_i g + \sum_{s,t=1}^{n} \alpha_{st}^i e_{st} = a_i g - \sum_{t=1}^{n} a_t e_{it}$. $\square$

2.3. **Yetter-Drinfeld modules over the dual of a group algebra.** Let H be a finite-dimensional Hopf algebra and ${}^H_H\mathcal{YD}$ be the category of left Yetter-Drinfeld modules over H, see e.g. [M]. The category ${}^{H^*}_{H^*}\mathcal{YD}$ is braided equivalent to ${}^H_H\mathcal{YD}$ by the following recipe. Let (h_i) and (f_i) be dual basis of H and H^*. Let $V \in {}^H_H\mathcal{YD}$, $v \in V$ and $f \in H^*$, then V turns into a Yetter-Drinfeld module over H^* by

$$f \cdot v = \langle \mathcal{S}(f), v_{(-1)}\rangle v_{(0)}, \quad \lambda(v) = \sum_i \mathcal{S}^{-1}(f_i) \otimes h_i \cdot v, \quad f \in H^*, v \in V. \tag{2.2}$$

This gives a braided equivalence between ${}^H_H\mathcal{YD}$ and ${}^{H^*}_{H^*}\mathcal{YD}$ [AG1, 2.2.1].

Let G be a finite group. If $g \in G$, then we denote by $\mathcal{O}_g$ the conjugacy class of g and by $C_G(g)$ the centralizer of g.

Definition 2.2. Fix $g \in G$ and (ρ, V) an irreducible representation of $C_G(g)$. Then

$$M(g,\rho) := \operatorname{Ind}_{C_G(g)}^G V = \Bbbk G \otimes_{C_G(g)} V = \mathcal{O}_g \otimes V$$

is an object of ${}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$ in the following way. Let $(w_i)_{1\le i\le r}$ be a basis of V and $(h_j)_{1\le j\le s}$ be a set of representatives of $G/C_G(g)$, so that $(h_j \otimes w_i)_{ji}$ is a basis of $M(g,\rho)$. Let $t_j = h_j g h_j^{-1}$, $1 \le j \le s$. The action and coaction on $M(g,\rho)$ are given by

$$h \cdot (h_j \otimes w_i) = h_k \otimes \rho(\tilde{g})(w_i) \qquad \text{and} \qquad \delta(h_j \otimes w_i) = t_j \otimes (h_j \otimes w_i),$$

$1 \le i \le r$, $1 \le j \le s$ and $h \in G$, where $hh_j = h_k\tilde{g}$ for unique k, $\tilde{g} \in C_G(g)$.

Let $\mathcal{Q}$ be a set of representatives of conjugacy class of G. It is well-known that $M(g,\rho)$ is simple and that any simple object of ${}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$ is isomorphic to $M(g,\rho)$ for unique $g \in \mathcal{Q}$ and a unique isomorphism class $[(\rho, V)]$ of irreducible representations of $C_G(g)$. By the braided equivalence described above, see (2.2), this gives a parametrization of the irreducible objects in ${}^{\Bbbk^G}_{\Bbbk^G}\mathcal{YD}$. Explicitly, $M(g,\rho) \in {}^{\Bbbk^G}_{\Bbbk^G}\mathcal{YD}$ with

$$\delta_h \cdot (h_j \otimes w_l) = \delta_{h,t_j^{-1}} h_j \otimes w_l,\ \lambda(h_j \otimes w_l) = \sum_{h\in G} \delta_{h^{-1}} \otimes h \cdot (h_j \otimes w_l). \tag{2.3}$$

2.4. **Nichols algebras.** Our reference for Nichols algebras is [AnS2, Subsection 2.1]. Let $V \in {}^H_H\mathcal{YD}$. The Nichols algebra $\mathfrak{B}(V) = \oplus_{n\in\mathbb{N}_0}\mathfrak{B}^n(V)$ of V is the braided graded Hopf algebra in ${}^H_H\mathcal{YD}$ which satisfies:

- $\mathfrak{B}^0(V) = \Bbbk 1$ and $\mathcal{P}(\mathfrak{B}(V)) = \mathfrak{B}^1(V) \simeq V$ in ${}^H_H\mathcal{YD}$.
- $\mathfrak{B}(V)$ is generated as an algebra by $\mathfrak{B}^1(V)$.

The tensor algebra $T(V)$ is a braided graded Hopf algebra in ${}^H_H\mathcal{YD}$ such that $\Delta(v) = v \otimes 1 + 1 \otimes v$ and $\varepsilon(v) = 0$ for all $v \in V$. Then the Nichols algebra can be described as $\mathfrak{B}(V) = T(V)/\mathcal{J}$, where $\mathcal{J}$ is the largest Hopf ideal of $T(V)$ generated by homogeneous elements of degree ≥ 2.

Lemma 2.3. *If $x \in \mathcal{J}^2 = \mathcal{J} \cap V^{\otimes 2}$, then $x \in \mathcal{P}(T(V))$.*

Proof. We have $\Delta_{T(V)}(x) = x \otimes 1 + 1 \otimes x + y$ with $y \in V \otimes V \cap (\mathcal{J} \otimes T(V) + T(V) \otimes \mathcal{J})$. Since $\mathcal{J} \subset \oplus_{n\ge 2} V^{\otimes n}$, $y = 0$. $\square$

2.5. **Hopf algebras whose coradical is a Hopf subalgebra.** Let H be a semisimple and cosemisimple Hopf algebra (here char $\Bbbk = 0$ is not needed). Let A be a Hopf algebra whose coradical A_0 is a Hopf subalgebra isomorphic to H. By [M, 5.2.8], the coalgebra $\operatorname{gr} A$ becomes a graded Hopf algebra. The projection $\pi : \operatorname{gr} A \to H$ with kernel $\oplus_{n>0} \operatorname{gr}^n A$ splits the inclusion $\iota : H \to \operatorname{gr} A$, i. e. $\pi\circ\iota = \mathrm{id}_H$. Then $R = \operatorname{gr} A^{\operatorname{co}\pi} = \{a \in \operatorname{gr} A | (\mathrm{id}\otimes\pi)\Delta(a) = a\otimes 1\}$ is a braided Hopf algebra in ${}^H_H\mathcal{YD}$ and $\operatorname{gr} A \simeq R\#H$ as Hopf algebras [R, Ma].

Here # stands for the smash product and the smash coproduct structures. Moreover, $\operatorname{gr} A$ and R have the following properties, see [AnS1]:

(a) $R = \oplus_{n\geq 0} R^n$ is a braided graded Hopf algebra with $R^n = R \cap \operatorname{gr}^n A$.
(b) $\operatorname{gr}^n A = R^n \# H$.
(c) $R_0 = R^0 = \Bbbk 1$ and $R^1 = P(R)$.

Clearly, $A \in {}_H\mathcal{M}_H$ by left and right multiplication. By [ArMŞ, Thm. 5.9.c)], there exists a projection $\Pi : A \to H$ of H-bimodule coalgebras such that $\Pi_{|H} = \mathrm{id}_H$. Hence A is a Hopf bimodule over H with coactions $\rho_L := (\Pi \circ \mathrm{id})\Delta$ and $\rho_R := (\mathrm{id} \circ \Pi)\Delta$. Thus by [AN, Lemma 1.1], $A_1 = H \oplus P_1$ as Hopf bimodules over H where

$$P_1 := A_1 \cap \ker \Pi = \{a \in A | \Delta(a) = \rho_L(a) + \rho_R(a)\}. \tag{2.4}$$

Let $V = R^1 \in {}^H_H\mathcal{YD}$; V is called the *infinitesimal braiding* of A. There exists an isomorphism $\gamma : V\#H \to P_1$ of Hopf bimodules over H, by (b) above.

Proposition 2.4. *Let A be a Hopf algebra whose coradical is a Hopf subalgebra isomorphic to H. Then $\operatorname{gr} A \simeq \mathfrak{B}(V)\#H$ if and only if there exists an epimorphism of Hopf algebras $\phi : T(V)\#H \to A$ such that $\phi_{|H} = \mathrm{id}_H$ and $\phi_{|V\#H} : V\#H \to P_1$ is an isomorphism of Hopf bimodules over H.*

Proof. Suppose first that $\operatorname{gr} A \simeq \mathfrak{B}(V)\#H$. Let $\gamma : V\#H \to P_1$ be an isomorphism of Hopf bimodules over H as above. Then $\phi : T(V)\#H \to A$ defined by $\phi(h) = h$ and $\phi(v) = \gamma(v\#1)$ is a morphism of Hopf algebras. Since A is generated by A_1 by [AnS1, Lemma 2.2], ϕ is an epimorphism.

Now, suppose that such a ϕ is given (for an arbitrary V). By [M, Cor. 5.3.5] $H = A_0$; hence $\operatorname{gr} A \simeq R\#H$. Since $\phi_{|V\#H} : V\#H \to P_1$ and $\gamma : R^1\#H \to P_1$ are isomorphisms of Hopf bimodules over H, $\mathcal{P}(R) = R^1 \simeq V$ in ${}^H_H\mathcal{YD}$. By the definition of Nichols algebra of V, we are left to show that V generates R but this follows from the fact that ϕ is surjective. $\square$

Remark 2.5. Let $\mathrm{ad} : A \to \operatorname{End} A$ be the adjoint representation, $\mathrm{ad}\, x(y) = x_{(1)} y \mathcal{S}(x_{(2)})$, $x, y \in A$. In the notation of Proposition 2.4, $\phi(h \cdot x) = \mathrm{ad}\, h(x)$ for all $h \in H$ and $x \in T(V)$. If $\mathcal{J}$ is the defining ideal of $\mathfrak{B}(V)$ as above, then $\phi(\mathcal{J}^2\#1) \subset A_1$ by Lemma 2.3.

3. Hopf algebras with coradical $\Bbbk^{\mathbb{S}_3}$

3.1. **Hopf algebras with coradical a dual group algebra.** We start discussing Hopf algebras A with coradical $\Bbbk^G$, G a finite group. Let $(\delta_h)_{h\in G}$ be the basis of $\Bbbk^G$ dual to the basis consisting of group-like elements of $\Bbbk G$; $\delta_h(g) = \delta_{g,h}$, where the last is the Dirac delta. Then for all $h, t \in G$,

$$\Delta(\delta_h) = \sum_{t\in G} \delta_t \otimes \delta_{t^{-1}h}, \qquad \mathrm{ad}\, \delta_h(\delta_t) = \delta_{h,e}\delta_t.$$

If $M \in {}_H\mathcal{M}$, then we set $M^g := \{x \in M | \delta_h \cdot x = \delta_{h,g} x \,\forall h \in G\}$, $g \in G$, and $\operatorname{supp} M := \{g \in G : M^g \neq 0\}$. This applies to the n-th term A_n of the coradical filtration, a $\Bbbk^G$-module via the adjoint action.

Lemma 3.1. *Let A be a Hopf algebra with coradical $\Bbbk^G$ and infinitesimal braiding $V = \oplus_{i\in I} M(g_i, \rho_i)$. Then*

(a) $A_n^g \cdot A_m^h \subseteq A_{n+m}^{gh}$ *for all* $n, m \geq 0$ *and* $g, h \in G$. *Hence* $A_n^g \in {}_{\Bbbk^G}\mathcal{M}_{\Bbbk^G}$.
(b) *If* $x_g \in A_n^g$ *then* $\delta_h x_g = x_g \delta_{g^{-1}h}$ *for all* $h \in G$.
(c) *If* $x_g \in A_n^g$, $g \in G$, *then* $\Delta(x_g) = \sum_{t\in G}(y_g^t \otimes \delta_t + \delta_t \otimes z_{t^{-1}gt}^t) + w$ *with* $w \in \bigoplus_{s,t\in G} \bigoplus_{i=1}^{n-1} (A_i^s \otimes A_{n-i}^t)$ *and* $y_g^t, z_g^t \in A_n^g$.
(d) *If* $g \in G$, *then* $\mathcal{S}(A_n^g) = A_n^{g^{-1}}$.
(e) $(\operatorname{supp} A_1)^{-1} = \bigcup_{i\in I} \mathcal{O}_{g_i} \cup \{e\}$.
(f) *If A is finite-dimensional then* $A_1^e = \Bbbk^G$.

Proof. Let $x_g \in A_n^g$ and $x_h \in A_m^h$ then

$$\operatorname{ad}\delta_s(x_g x_h) = \sum_{t\in G} \operatorname{ad}\delta_t(x_g) \operatorname{ad}\delta_{t^{-1}s}(x_h) = \delta_{s,gh} x_g x_h,$$

since the only non-zero term occurs when $t = g$ and $t^{-1}s = h$. This implies (a); note that A_n^g is a $\Bbbk^G$-bimodule because $\Bbbk^G = A_0^e$. Now (b) follows from

$$\delta_h x_g = \sum_{s\in G} \delta_s x_g \varepsilon(\delta_{s^{-1}h}) = \sum_{s\in G} \operatorname{ad}\delta_s(x_g)\delta_{s^{-1}h} = x_g \delta_{g^{-1}h}.$$

By (2.1), we can write $\Delta(x_g) = \sum_{s,t\in G} (y_s^t \otimes \delta_t + \delta_t \otimes z_s^t) + w$ with $y_s^t, z_s^t \in A_n^s$ and $w \in \bigoplus_{\substack{s,t\in G\\ 1\leq i\leq n-1}} (A_i^s \otimes A_{n-i}^t)$. If $w = w_1 \otimes w_2$, then $\tilde{w} = \sum_{f,h,s,t\in G} \delta_f w_1 \mathcal{S}(\delta_{h^{-1}g}) \otimes \operatorname{ad}\delta_{f^{-1}h}(w_2)$ also belongs to $\bigoplus_{\substack{s,t\in G\\ 1\leq i\leq n-1}} A_i^s \otimes A_{n-i}^t$. Then

$$\begin{aligned}
\Delta(x_g) = \Delta(\operatorname{ad}\delta_g(x_g)) &= \sum_{f,h,s,t\in G} \delta_f y_s^t \mathcal{S}(\delta_{h^{-1}g}) \otimes \operatorname{ad}\delta_{f^{-1}h}(\delta_t) \\
&+ \sum_{f,h,s,t\in G} \delta_f \delta_t \mathcal{S}(\delta_{h^{-1}g}) \otimes \operatorname{ad}\delta_{f^{-1}h}(z_s^t) + \tilde{w} \\
&= \sum_{h,s,t\in G} \delta_h y_s^t \mathcal{S}(\delta_{h^{-1}g}) \otimes \delta_t + \sum_{f,s,t\in G} \delta_f \delta_t \mathcal{S}(\delta_{(fs)^{-1}g}) \otimes z_s^t + \tilde{w} \\
&= \sum_{s,t\in G} \operatorname{ad}\delta_g(y_s^t) \otimes \delta_t + \sum_{s,t\in G} \delta_t \delta_{g^{-1}ts} \otimes z_s^t + \tilde{w} \\
&= \sum_{t\in G} y_g^t \otimes \delta_t + \sum_{t\in G} \delta_t \otimes z_{t^{-1}gt}^t + \tilde{w}.
\end{aligned}$$

The proof of (d) is straightforward. We observed in Subsection 2.5 that $A_1 = \Bbbk^G \oplus V\#\Bbbk^G$ as Hopf bimodules. Thus $A_1^g = V^g\#\Bbbk^G$, $g \neq e$, and $A_1^e = \Bbbk^G \oplus V^e\#\Bbbk^G$. By (2.3), (e) follows.

Let K be the subalgebra generated by A_1^e; by (c) A_1^e is a coalgebra and by (d) it is stable under the antipode, hence K is a Hopf subalgebra. By (b) $\Bbbk^G$ is a normal Hopf subalgebra of K and if $\dim A < \infty$, then we have

the exact sequence of Hopf algebras $\Bbbk^G \hookrightarrow K \twoheadrightarrow K/(\Bbbk^G)^+K$, see [AD]. We claim that $K/(\Bbbk^G)^+K = \Bbbk$ and therefore $\Bbbk^G = K$ by [AD].

Let $x_e \in B := V^e \# \Bbbk^G$, identified with a subspace of A_1^e as above. Since $V\#\Bbbk^G \simeq P_1 \subset \ker\varepsilon$, see (2.4), we have

$$A_1^e = \Bbbk\delta_e \oplus (\sum_{t\neq e} \Bbbk\delta_t) \oplus B \text{ and } A_1^e \cap \ker\varepsilon = \sum_{t\neq e} \Bbbk\delta_t \oplus B.$$

By (c), $\Delta(x_e) \in A_1^e \otimes \Bbbk^G + \Bbbk^G \otimes A_1^e = (B \otimes \Bbbk^G) \oplus (\Bbbk^G \otimes B) \oplus (\Bbbk^G \otimes \Bbbk^G)$. Write correspondingly

$$\Delta(x_e) = \sum_{t\in G}(u_t \otimes \delta_t + \delta_t \otimes v_t) + \sum_{s,t\in G} b_{s,t}\delta_s \otimes \delta_t,$$

with $u_t, v_t \in B$ and $b_{s,t} \in \Bbbk$. Computing $(\varepsilon \otimes \mathrm{id})\Delta(x_e)$ and $(\mathrm{id} \otimes\varepsilon)\Delta(x_e)$, we get $x_e = v_e + \sum_{s\in G} b_{e,s}\delta_s = u_e + \sum_{s\in G} b_{s,e}$ and then $v_e = x_e = u_e$ and $b_{e,s} = 0 = b_{s,e}$ for all $s \in G$. Hence $\overline{x_e}$ is a primitive element in $K/(\Bbbk^G)^+K$, therefore $\overline{x_e} = 0$ since $\dim A < \infty$ and the claim is true. $\square$

3.2. **Hopf algebras with coradical $\Bbbk^{\mathbb{S}_n}$.** Let $\mathcal{O}_2^n$ be the conjugacy class of (12) in $\mathbb{S}_n$ and let $\mathrm{sgn} : C_{\mathbb{S}_n}(12) \to \Bbbk$ be the restriction of the sign representation of $\mathbb{S}_n$. Let $V_n = M((12), \mathrm{sgn}) \in {}^{\Bbbk\mathbb{S}_n}_{\Bbbk\mathbb{S}_n}\mathcal{YD}$, cf. Definition 2.2; V_n has a basis $(x_{(ij)})_{(ij)\in\mathcal{O}_2^n}$ such that $\delta(x_{(ij)}) = (ij) \otimes x_{(ij)}$ and $g \cdot x_{(ij)} = \mathrm{sgn}(g)x_{g(ij)g^{-1}}$, $g \in \mathbb{S}_n$. By (2.3), V_n turns into an object in ${}^{\Bbbk^{\mathbb{S}_n}}_{\Bbbk^{\mathbb{S}_n}}\mathcal{YD}$ with action and coaction given by

$$\delta_h \cdot x_{(ij)} = \delta_{h,(ij)}\, x_{(ij)} \qquad \text{and} \qquad \lambda(x_{(ij)}) = \sum_{h\in G} \mathrm{sgn}(h)\delta_h \otimes x_{h^{-1}(ij)h}.$$

Let $\mathcal{J}_n$ be the ideal of relations of $\mathfrak{B}(V_n)$. The elements

$$x^2_{(ij)}, \tag{3.1}$$

$$R_{(ij)(kl)} := x_{(ij)}x_{(kl)} + x_{(kl)}x_{(ij)}, \tag{3.2}$$

$$R_{(ij)(ik)} := x_{(ij)}x_{(ik)} + x_{(ik)}x_{(jk)} + x_{(jk)}x_{(ij)} \tag{3.3}$$

for all $(ij), (lk), (ik) \in \mathcal{O}_2^n$ such that $(lk) \neq (ij) \neq (ik)$ form a basis of $\mathcal{J}_n^2$.

Remark 3.2. It was shown by [MS] for $n = 3, 4$ and by [G] for $n = 5$ that $\mathcal{J}_n$ is generated by these elements and $\mathfrak{B}(V_n)$ is finite-dimensional.

We first show that relations (3.2) and (3.3) hold in any lifting of $\mathfrak{B}(V_n)$. In what follows, A is a Hopf algebra such that $\mathrm{gr}\, A \simeq \mathfrak{B}(V_n)\#\Bbbk^{\mathbb{S}_n}$ and $\phi : T(V_n)\#\Bbbk^{\mathbb{S}_n} \to A$ is as in Proposition 2.4.

Lemma 3.3. *We have $\phi(R_{(ij)(lk)}) = 0$ for all $(ij) \neq (kl) \in \mathcal{O}_2^n$. If A has finite dimension, then $\phi(\sum_{(ij)\in\mathcal{O}_2^n} x^2_{(ij)}) = 0$.*

Proof. Since $\phi(R_{(ij)(kl)}) \in A_1^{(ij)(kl)}$ by Lemma 2.3, Remark 2.5 and Lemma 3.1 (a), we see that $\phi(R_{(ij)(lk)}) = 0$ from Lemma 3.1 (e). Since $\sum_{(ij)\in\mathcal{O}_2^n} x^2_{(ij)}$

is primitive in $T(V)$ and spans the trivial Yetter-Drinfeld module, we conclude that $\phi(\sum_{(ij)\in\mathcal{O}_2^n} x_{(ij)}^2)$ is primitive in A, hence it is 0. □

Definition 3.4. Given $(a_1, a_2) \in \Bbbk^2$, we denote by $\mathcal{A}_{[a_1,a_2]}$ the algebra $(T(V_3)\#\Bbbk^{\mathbb{S}_3})/\mathcal{I}_{(a_1,a_2)}$ where $\mathcal{I}_{(a_1,a_2)}$ is the ideal generated by

$$\begin{aligned}
&R_{(13)(23)},\\
&R_{(23)(13)},\\
&x_{(13)}^2 - (a_1 - a_2)(\delta_{(12)} + \delta_{(123)}) - a_1(\delta_{(23)} + \delta_{(132)}),\\
&x_{(23)}^2 - a_2(\delta_{(13)} + \delta_{(123)}) - (a_2 - a_1)(\delta_{(12)} + \delta_{(132)}),\\
&x_{(12)}^2 + a_1(\delta_{(23)} + \delta_{(123)}) + a_2(\delta_{(13)} + \delta_{(132)}).
\end{aligned} \tag{3.4}$$

It is easy to see that $\mathcal{I}_{(a_1,a_2)}$ is a Hopf ideal, hence $\mathcal{A}_{[a_1,a_2]}$ is a Hopf algebra. Clearly, $\mathcal{A}_{[0,0]} \simeq \mathfrak{B}(V_3)\#\Bbbk^{\mathbb{S}_3}$.

Recall that $\Gamma = \Bbbk^\times \times \mathbb{S}_3$ acts on $\Bbbk^2$ by (1.1). Here is the main result of this paper. Notice that Lemma 2.1 is used in the proof to find out the deformed relations.

Theorem 3.5. *(a) Let A be a finite-dimensional non-semisimple Hopf algebra with coradical $\Bbbk^{\mathbb{S}_3}$. Then $A \simeq \mathcal{A}_{[a_1,a_2]}$ for some $(a_1, a_2) \in \Bbbk^2$.*

(b) The coradical of $\mathcal{A}_{[a_1,a_2]}$ is isomorphic to $\Bbbk^{\mathbb{S}_3}$ and $\dim \mathcal{A}_{[a_1,a_2]} = 72$ *for all $(a_1, a_2) \in \Bbbk^2$.*

(c) $\mathcal{A}_{[b_1,b_2]} \simeq \mathcal{A}_{[a_1,a_2]}$ if and only if $[b_1, b_2] = [a_1, a_2]$.

Proof. (a) If $\operatorname{gr} A \simeq R\#\Bbbk^{\mathbb{S}_3}$, then $V = R^1 \in {}^{\Bbbk^{\mathbb{S}_3}}_{\Bbbk^{\mathbb{S}_3}}\mathcal{YD}$ and $\mathfrak{B}(V)$ is a braided Hopf subalgebra of R by [AnS2, Prop. 2.2]. Thus $V \simeq V_3$ because $\mathfrak{B}(V_3)$ is the only Nichols algebra of finite dimension in ${}^{\Bbbk\mathbb{S}_3}_{\Bbbk\mathbb{S}_3}\mathcal{YD}$ by [AHS, Thm. 4.5]. Since $\mathfrak{B}(V_3)$ only depends on the braiding of V, we can deduce from [AG2, Thm. 2.1] that $R = \mathfrak{B}(V_3)$.

Let $\rho : \mathbb{S}_3 \to GL(2, \Bbbk)$ be the irreducible representation defined in the canonical basis (e_i) by

$$\rho(12) \cdot e_1 = e_2, \qquad \rho(123) \cdot e_1 = -e_2.$$

Let f_{ji} be the matrix coefficients of ρ; thus

$$\begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} = \begin{pmatrix} \delta_e + \delta_{(13)} - \delta_{(23)} - \delta_{(132)} & \delta_{(12)} - \delta_{(23)} + \delta_{(123)} - \delta_{(132)} \\ \delta_{(12)} - \delta_{(13)} - \delta_{(123)} + \delta_{(132)} & \delta_e - \delta_{(13)} + \delta_{(23)} - \delta_{(123)} \end{pmatrix}$$

Considering $T(V_3) \in {}^{\Bbbk\mathbb{S}_3}_{\Bbbk\mathbb{S}_3}\mathcal{YD}$, we see that the elements $c_1 := x_{(13)}^2 - x_{(12)}^2$ and $c_2 := x_{(23)}^2 - x_{(12)}^2$ of $T(V)$ span a Yetter-Drinfeld submodule isomorphic to $M(e, \rho)$, via $e_i \mapsto c_i$, $1 \le i \le 2$. Hence this subspace becomes a submodule

in ${}^{\Bbbk^{\mathbb{S}_3}}_{\Bbbk^{\mathbb{S}_3}}\mathcal{YD}$, with coaction λ given by $\lambda(c_i) = \sum_{j=1}^{2} e_{ij} \otimes c_j$, where $e_{ij} := \mathcal{S}^{-1}(f_{ji})$ for $i, j = 1, 2$. Then

$$\Delta\phi(c_i) = \phi(c_i) \otimes 1 + \sum_j e_{ij} \otimes \phi(c_j).$$

Since $\phi(c_1)$, $\phi(c_2) \in A_1^e = \Bbbk^{\mathbb{S}_3}$, we conclude that they belong to $\Bbbk 1 \oplus C$, where C is the simple subcoalgebra of rank 2 spanned the e_{ij}'s. By Lemma 2.1, there exist $a_1, a_2 \in \Bbbk$ such that $\phi(c_i) = a_i - \sum_{j=1}^{2} a_j e_{ij}$ for $i = 1, 2$. Note that the elements (3.4) form a basis of

$$M_{(a_1,a_2)} = \langle c_i - a_i + \sum_j a_j e_{ij},\, R_{(13)(23)},\, R_{(23)(13)},\, \sum_{(ij)\in\mathcal{O}_2^3} x_{(ij)}^2 \rangle,$$

which is a coideal contained in $\ker \varepsilon$. Then the ideal $\mathcal{I}_{(a_1,a_2)}$ generated by $M_{(a_1,a_2)}$ is a Hopf ideal and $\mathcal{I}_{(a_1,a_2)} \cap (\Bbbk \oplus V_3)\#\Bbbk^{\mathbb{S}_3} = 0$ because $\mathcal{I}_{(a_1,a_2)} \subset \ker \phi$. Thus $\operatorname{gr}(T(V_3)\#\Bbbk^{\mathbb{S}_3}/\mathcal{I}_{(a_1,a_2)}) \simeq R\#\Bbbk^{\mathbb{S}_3}$ and R is generated by $\phi(V_3)$. Hence $R \simeq T(V_3)/I$ with $I \subseteq \mathcal{J}_3$. Moreover, the generators of $\mathcal{J}_3$ are contained in I by the definition of $\mathcal{I}_{(a_1,a_2)}$ and therefore $I = \mathcal{J}_3$. Then $\dim A = \dim(T(V_3)\#\Bbbk^{\mathbb{S}_3}/\mathcal{I}_{(a_1,a_2)})$, that is, $\ker \phi = \mathcal{I}_{(a_1,a_2)}$.

We claim that $\mathcal{B} = \{x\delta_g | x \in B,\, g \in \mathbb{S}_3\}$ is a basis of $\mathcal{A}_{[a_1,a_2]}$ where

$$B = \left\{ \begin{matrix} 1, & x_{(13)}, & x_{(13)}x_{(12)}, & x_{(13)}x_{(12)}x_{(13)}, & x_{(13)}x_{(12)}x_{(23)}x_{(12)}, \\ & x_{(23)}, & x_{(12)}x_{(13)}, & x_{(12)}x_{(23)}x_{(12)}, & \\ & x_{(12)}, & x_{(23)}x_{(12)}, & x_{(13)}x_{(12)}x_{(23)}, & \\ & & x_{(12)}x_{(23)} & & \end{matrix} \right\}$$

and therefore (b) follows. Next, we sketch a proof of the claim using the Diamond Lemma [B].

We need to introduce more relations which are deduced from (3.4). We write the relations of the form $R = f$ with R a monomial of $\mathcal{A}_{[a_1,a_2]}$ and $f \in \Bbbk\mathcal{B}$. The new list of relations is

$$1 = \sum_{g\in\mathbb{S}_3} \delta_g, \quad \delta_g\delta_h = \delta_{g,h}\delta_g, \quad \delta_g x_{(ij)} = x_{(ij)}\delta_{(ij)g},$$

$$\begin{aligned} x_{(13)}^2 =& (a_1 - a_2)(\delta_{(12)} + \delta_{(123)}) + a_1(\delta_{(23)} + \delta_{(132)}), \\ x_{(23)}^2 =& a_2(\delta_{(13)} + \delta_{(123)}) + (a_2 - a_1)(\delta_{(12)} + \delta_{(132)}), \\ x_{(12)}^2 =& -a_1(\delta_{(23)} + \delta_{(123)}) - a_2(\delta_{(13)} + \delta_{(132)}), \\ x_{(13)}x_{(23)} =& -x_{(23)}x_{(12)} - x_{(12)}x_{(13)}, \\ x_{(23)}x_{(13)} =& -x_{(12)}x_{(23)} - x_{(13)}x_{(12)}, \\ x_{(12)}x_{(13)}x_{(12)} =& x_{(13)}x_{(12)}x_{(13)} + x_{(23)}a_1, \\ x_{(23)}x_{(12)}x_{(23)} =& x_{(12)}x_{(23)}x_{(12)} - x_{(13)}a_2 \quad \text{and} \\ x_{(23)}x_{(12)}x_{(13)} =& x_{(13)}x_{(12)}x_{(23)} + x_{(12)}\Omega \end{aligned}$$

where $\Omega = (a_2-a_1)(\delta_{(12)}-\delta_e)+a_1(\delta_{(13)}-\delta_{(132)})-a_2(\delta_{(23)}-\delta_{(123)})$. Following the Diamond Lemma, we have to show that if X, Y, Z are monomial in $A\backslash\{1\}$ such that $R_1 = XY$ and $R_2 = YZ$ then f_1Z and Xf_2 can be reduced to a same element in $\Bbbk\mathcal{B}$ using the before list of relations – XYZ is called "overlap ambiguity" and if the above is true it said that the ambiguity is "resoluble"; it is defined in [B] other type of ambiguity but this does not happen in our case. Calculate the ambiguities and show that these are resoluble is a tedious but straightforward computation.

(c) We denote the elements in $\mathcal{A}_{[a_1,a_2]}$ and $\mathcal{A}_{[b_1,b_2]}$ by ${}^a\overline{x}\delta_g$ and ${}^b\overline{x}\delta_g$ for $x \in T(V_3)$, $g \in \mathbb{S}_3$. Let $\Theta : \mathcal{A}_{[b_1,b_2]} \to \mathcal{A}_{[a_1,a_2]}$ be an isomorphism of Hopf algebras. Since $(\Theta_{|\Bbbk^{\mathbb{S}_3}})^*$ induces a group automorphism of $\mathbb{S}_3$, $\Theta(\delta_g) = \delta_{\theta g\theta^{-1}}$ for some $\theta \in \mathbb{S}_3$. By the adjoint action of $\Bbbk^{\mathbb{S}_3}$, $\Theta({}^b\overline{x_{(ij)}}) \in \sum_{g\in\mathbb{S}_3} \lambda_g {}^a\overline{x_{\theta(ij)\theta^{-1}}}\delta_g$ with $\lambda_g \in \Bbbk$ but since Θ is a coalgebra morphism, all these λ_g's are equal. Then for each $(ij) \in \mathcal{O}_2^3$ there exists $\mu_{(ij)} \in \Bbbk^\times$ such that $\Theta({}^b\overline{x_{(ij)}}) = \mu_{(ij)} {}^a\overline{x_{\theta(ij)\theta^{-1}}}$. Also,

$$0 = {}^a\overline{\mu_{(23)}\mu_{(13)}x_{(23)}x_{(13)} + \mu_{(13)}\mu_{(12)}x_{(13)}x_{(12)} + \mu_{(12)}\mu_{(23)}x_{(12)}x_{(23)}}$$

because it is equal to $\Theta({}^b\overline{R_{(13)(23)}})$ if $\theta = (12)$ or to $\Theta({}^b\overline{R_{(23)(13)}})$ if $\theta = (123)$. Then $\mu_{(23)}\mu_{(13)} = \mu_{(13)}\mu_{(12)} = \mu_{(12)}\mu_{(23)}$ because otherwise $\dim \mathcal{A}(a_1, a_2)$ would be less than 72. Since $\mu_{(ij)} \neq 0$ it results that $\Theta({}^b\overline{x_{(ij)}}) = \mu\, {}^a\overline{x_{\theta(ij)\theta^{-1}}}$ with $\mu \in \Bbbk^\times$ for all $(ij) \in \mathcal{O}_2^3$. Since $(\delta_g)_{g\in\mathbb{S}_3}$ is linearly independent, we obtain that $(b_1, b_2) = (a_1, a_2) \triangleleft (\mu^2, \theta)$ by the equality $\Theta({}^b\overline{x_{(12)}^2}) = {}^a\overline{x_{\theta(12)\theta^{-1}}^2}$. Conversely, given $(\mu, \theta) \in \Gamma$, the map $\Theta_{\mu,\theta} : T(V_3)\#\Bbbk^{\mathbb{S}_3} \to T(V_3)\#\Bbbk^{\mathbb{S}_3}$ defined by $\Theta_{\mu,\theta}(\delta_g) = \delta_{\theta g\theta^{-1}}$, $\Theta_{\mu,\theta}(x_{(ij)}) = \mu x_{\theta(ij)\theta^{-1}}$, for $g \in \mathbb{S}_3$. $(ij) \in \mathcal{O}_2^3$, is a Hopf algebra isomorphism such that $\Theta_{\mu,\theta}(\mathcal{I}_{(a_1,a_2)}) = \mathcal{I}_{(a_1,a_2)\triangleleft(\mu^2,\theta)^{-1}}$. $\square$

References

[AD] Andruskiewitsch N., Devoto J., Extensions of Hopf algebras, *Algebra i Analiz* **7**, No. 1 (1995), 22-61.

[AG1] Andruskiewitsch, N., Graña, M. , Braided Hopf algebras over non abelian finite group, *Bol. Acad. Ciencias (Córdoba)* **63** (1999), 45-78.

[AG2] Andruskiewitsch, N., Graña, M. , Examples of liftings of Nichols algebras over racks, *Algebra Montp. Announc.* 2003, Paper 1, 6 pp. (electronic).

[AHS] N. Andruskiewitsch, I. Heckenberger and H.-J. Schneider, *The Nichols algebra of a semisimple Yetter-Drinfeld module*, Amer. J. Math. **132**, (2010).

[AN] Andruskiewitsch N., Natale S., Counting arguments for Hopf algebras of low dimension, *Tsukuba Math J.* **25**, No. 1 (2001), 187-201.

[AnS1] Andruskiewitsch, N. , Schneider, H.-J. , Lifting of Quantum Linear Space and Pointed Hopf algebras of order p^3, *J. Algebra* **209** (1998), no. 2, 658-691.

[AnS2] Andruskiewitsch, N. , Schneider, H.-J. , Pointed Hopf Algebras, in "New directions in Hopf algebras", *Math. Sci. Res. Inst. Publ.*, Cambridge Univ. Press, Cambridge, **43** (2002), 1-68.

[ArMŞ] Ardizzoni A., Menini C. and Ştefan D., A Monoidal Approach to Splitting Morphisms of Bialgebras, *Trans. Amer. Math. Soc.* **359** (2007), 991-1044.

[B] Bergman, G. M., The Diamond Lemma for Ring Theory, *Adv. in Math.* **29** (1978), 178-218.

[CDMM] Calinescu C., Dăscălescu S., Masuoka A., Menini C., Quantum lines over non-cocommutative cosemisimple Hopf algebras, *J. Algebra* **273** (2004), 753-779.

[G] Graña, M. , Zoo of finite-dimensional Nichols algebras of non-abelian group type, available at `http://mate.dm.uba.ar/~matiasg/zoo.html`.

[Ma] S. Majid, Crossed products of braided groups and bosonization. *J. Algebra* **163** (1994), 165-190.

[M] Montgomery S., Hopf Algebras and their Actions on Rings, CBMS Reg. Conf. Ser. Math. 82, *Amer. Math. Soc.* (1993).

[MS] Milinski, A., Schneider H-J., Pointed indecomposable Hopf algebras over Coxeter groups, in "New Trends in Hopf algebras Theory", *Contemp. Math.* **267** (2000), 215-236.

[R] Radford, D., Hopf algebras with projection, *J. Algebra* **92** (1985), 322-347.

FaMAF-CIEM (CONICET), Universidad Nacional de Córdoba, Medina Allende s/n, Ciudad Universitaria, 5000 Córdoba, República Argentina.

E-mail address: `andrus@famaf.unc.edu.ar, vay@famaf.unc.edu.ar`